

# A non-unital algebra has UUNP iff its unitization has UUNP

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**Abstract.** Let  $A$  be a non-unital Banach algebra, S. J. Bhatt and H. V. Dedania showed that  $A$  has the unique uniform norm property (UUNP) if and only if its unitization has UUNP. Here we prove this result for any non-unital algebra.

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Let  $A$  be a non-unital algebra and let  $A_e = \{a + \lambda e : a \in A, \lambda \in \mathbb{C}\}$  be the unitization of  $A$  with the identity denoted by  $e$ . For an algebra norm  $\|\cdot\|$  on  $A$ , define  $\|a + \lambda e\|_{op} = \sup\{\|(a + \lambda e)b\| : b \in A, \|b\| \leq 1\}$  and  $\|a + \lambda e\|_1 = \|a\| + |\lambda|$  for all  $a + \lambda e \in A_e$ .  $\|\cdot\|_{op}$  is an algebra seminorm on  $A_e$ , and  $\|\cdot\|_1$  is an algebra norm on  $A_e$ . An algebra norm  $\|\cdot\|$  on  $A$  is called regular if  $\|\cdot\|_{op} = \|\cdot\|$  on  $A$ . A uniform norm  $\|\cdot\|$  on  $A$  is an algebra norm satisfying the square property  $\|a^2\| = \|a\|^2$  for all  $a \in A$ ; and in this case,  $\|\cdot\|$  is regular and  $\|\cdot\|_{op}$  is a uniform norm on  $A_e$ . An algebra has the unique uniform norm property (UUNP) if it admits exactly one uniform norm.

**Theorem .** A non-unital algebra  $A$  has UUNP if and only if its unitization  $A_e$  has UUNP.

Proof: Let  $\|\cdot\|$  and  $|||\cdot|||$  be two uniform norms on  $A_e$ , then  $\|\cdot\| = |||\cdot|||$  on  $A$  since  $A$  has UUNP, and so  $\|\cdot\|_{op} = |||\cdot|||_{op}$  on  $A_e$ . By [3, Corollary 2.2(1)] and since two equivalent uniform norms are identical, it follows that  $(\|\cdot\| = \|\cdot\|_{op} \text{ or } \|\cdot\| \cong \|\cdot\|_1)$  and  $(|||\cdot||| = |||\cdot|||_{op} = \|\cdot\|_{op} \text{ or } |||\cdot||| \cong |||\cdot|||_1 = \|\cdot\|_1)$ ; equivalently, at least one of the following holds:

- (i)  $\|\cdot\| = \|\cdot\|_{op}$  and  $|||\cdot||| = |||\cdot|||_{op} = \|\cdot\|_{op}$ ;
- (ii)  $\|\cdot\| = \|\cdot\|_{op}$  and  $|||\cdot||| \cong |||\cdot|||_1 = \|\cdot\|_1$ ;
- (iii)  $\|\cdot\| \cong \|\cdot\|_1$  and  $|||\cdot||| = |||\cdot|||_{op} = \|\cdot\|_{op}$ ;
- (iv)  $\|\cdot\| \cong \|\cdot\|_1$  and  $|||\cdot||| \cong |||\cdot|||_1 = \|\cdot\|_1$ .

If either (i) or (iv) is satisfied, then  $\|\cdot\| = |||\cdot|||$ . By noting that (ii) and (iii) are similar by interchanging the roles of  $\|\cdot\|$  and  $|||\cdot|||$ , it is enough to assume (ii). Let  $(c(A), \|\cdot\|^\sim)$  be the completion of  $(A, \|\cdot\|)$ , we distinguish two cases:

(1)  $c(A)$  has not an identity:

$\|\cdot\|^\sim$  is regular since it is uniform. By [1, Corollary 2],  $\|\cdot\|_{op}^\sim \leq \|\cdot\|_1^\sim \leq 3\|\cdot\|_{op}^\sim$  on  $c(A)_e$  (unitization of  $c(A)$ ). Let  $a + \lambda e \in A_e \subset c(A)_e$ ,  $\|a + \lambda e\|_1^\sim = \|a\|^\sim + |\lambda| = \|a\| + |\lambda| = \|a + \lambda e\|_1$  and  $\|a + \lambda e\|_{op}^\sim = \sup\{\|(a + \lambda e)b\|^\sim : b \in c(A), \|b\|^\sim \leq 1\} = \sup\{\|(a + \lambda e)b\| : b \in A, \|b\| \leq 1\} = \|a + \lambda e\|_{op}$ . Therefore  $\|\cdot\|_{op} \leq \|\cdot\|_1 \leq 3\|\cdot\|_{op}$ .

By (ii),  $\|\cdot\|$  and  $|||\cdot|||$  are equivalent uniform norms, and so  $\|\cdot\| = |||\cdot|||$ .

(2)  $c(A)$  has an identity  $e$  :

Let  $(c(A_e), |||\cdot|||^\sim)$  be the completion of  $(A_e, |||\cdot|||)$ . Since  $\|\cdot\| = |||\cdot|||$  on  $A$ ,  $c(A)$  can be identified to the closure of  $A$  in  $(c(A_e), |||\cdot|||^\sim)$  so that  $\|\cdot\|^\sim = |||\cdot|||^\sim$  on  $c(A)$ . Let  $a + \lambda e \in A_e \subset c(A)$ ,

$$\begin{aligned} \|a + \lambda e\| &= \|a + \lambda e\|_{op} \text{ by (ii)} \\ &= \sup\{\|(a + \lambda e)b\| : b \in A, \|b\| \leq 1\} \\ &= \sup\{\|(a + \lambda e)b\|^\sim : b \in c(A), \|b\|^\sim \leq 1\} \\ &= \|a + \lambda e\|^\sim \text{ since } c(A) \text{ is unital} \\ &= |||a + \lambda e|||^\sim = |||a + \lambda e|||. \text{ Thus } \|\cdot\| = |||\cdot|||. \end{aligned}$$

Conversely, let  $\|\cdot\|$  and  $|||\cdot|||$  be two uniform norms on  $A$ , then  $\|\cdot\|_{op}$  and  $|||\cdot|||_{op}$  are uniform norms on  $A_e$ , hence  $\|\cdot\|_{op} = |||\cdot|||_{op}$  since  $A_e$  has UUNP. Therefore  $\|\cdot\| = \|\cdot\|_{op} = |||\cdot|||_{op} = |||\cdot|||$  on  $A$  since  $\|\cdot\|$  and  $|||\cdot|||$  are regular.

## References

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